

The University of Chicago

A METHOD OF CALCULATING FLUCTUATIONS

A DISSERTATION SUBMITTED TO THE
FACULTY OF THE DIVISION OF THE
BIOLOGICAL SCIENCES IN CANDIDACY FOR
THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF PHYSICS

1938

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RAYMOND ELICKSON

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A Method of Calculating Fluctuations

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A general method for the solution of fluctuation problems is described. A generating function for the differential equations that are satisfied by the probabilities in question is defined and the problem is reduced to the determination of the generating function. The general method for its determination is given and the results are applied to fluctuation problems in pair-production and radioactive disintegration. The method of finding the mean, mean square, and standard deviation from the generating function without reference to the form of the distribution itself is applied to the examples. The problem of fluctuations in chain and branch radioactive disintegrations is solved without restrictions on the size of the disintegration constants involved.

INTRODUCTION

IN experiments on radioactivity, cosmic rays, energy loss of electrons, etc., fluctuations play an important part. In some cases it is relatively

easy to take these fluctuations into account, for example, those that follow the Poisson law. The Poisson law can be derived from the more general binomial law¹ if the restriction is made

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¹ L. I. Schiff, Phys. Rev. **50**, 88 (1936).

that the probability, q , of the elementary event in question is so small that the product Nq remains finite as N becomes large, N being the total number of individuals under consideration.

For a large number of individuals that can fall in any one of a large number of divisions, the law can be derived directly, without reference to the binomial law, subject to the following more general conditions:²

1. The chance of falling in a division is the same for each individual.
2. The fact that an individual has fallen in a division does not affect the chance of other individuals falling therein.
3. The chance of an individual falling in it is the same for each division.

Although not a part of the general conditions, in radioactive problems the number of atoms of the parent substance must remain essentially constant throughout the experiment. If this is not the case, the probable number of disintegrations will be a function of the time and subject to fluctuations itself.

Examples of fluctuation problems that do not meet these conditions, and therefore are not governed by the Poisson law, are the number of particles in a cosmic-ray shower and chain or branch radioactive disintegration. In this paper a method is given for calculating fluctuations for some problems that are not subject to the Poisson law. The method is a very general one and can easily be extended to cover many similar problems. It is often possible to obtain the complete solution by using it, although there are some problems (e.g. Furry's Problem B) for which it is not possible to carry out all the steps without undue labor.

ILLUSTRATION OF THE METHOD

While the method to be used is very general, it is easier to explain it in connection with special examples. One such example is Furry's Problem A,³ which is an idealization of the problem of the fluctuations in the number of particles in cosmic-ray showers: The probability that in traversing thickness dt one particle is converted into two is just dt . If one particle enters a sheet of thickness t , what is the probability $P(n, t)$ that n particles will emerge? The probabilities in question satisfy

the system of differential equations

$$(d/dt)P(n, t) = -nP(n, t) + (n-1)P(n-1, t) \quad (1)$$

with the boundary conditions

$$P(1, 0) = 1; \quad P(n, 0) = 0, \quad n \neq 1. \quad (2)$$

These equations can be solved, as Furry has done, by successive integration and application of the boundary conditions, or one may define a generating function⁴

$$Q(x, t) = \sum_{n=0}^{\infty} x^n P(n, t) \quad (3)$$

and attempt to determine it. Since

$$(\partial/\partial x)Q(x, t) = \sum_{n=0}^{\infty} nx^{n-1}P(n, t)$$

it follows from Eqs. (1) and (3) that

$$(\partial/\partial t)Q(x, t) = (x^2 - x)(\partial/\partial x)Q(x, t) \quad (4)$$

with the boundary condition

$$Q(x, 0) = x. \quad (5)$$

Having determined $Q(x, t)$, $P(n, t)$ can be found by series expansion. Other quantities of interest can be found from $Q(x, t)$ without determining the P 's; thus the mean number

$$\langle n \rangle = \sum_{n=0}^{\infty} nP(n, t) = [(\partial/\partial x)Q(x, t)]_{x=1}, \quad (6)$$

and the mean square

$$\langle n^2 \rangle = \sum_{n=0}^{\infty} n^2 P(n, t) = [(\partial/\partial x)x(\partial/\partial x)Q(x, t)]_{x=1}. \quad (7)$$

The mean square can also be found from Eq. (6) and the relation

$$\begin{aligned} \langle n^2 \rangle - \langle n \rangle &= \sum_{n=0}^{\infty} n(n-1)P(n, t) \\ &= [(\partial^2/\partial x^2)Q(x, t)]_{x=1}. \end{aligned}$$

The standard deviation, probable error, and probable percentage error can be found from Eqs. (6) and (7). Thus, in cases where the series expansion for the $P(n, t)$ is difficult to carry out,

² D. J. Struik, *J. Math. Phys. M. I. T.* **9**, 151 (1930).

³ W. H. Furry, *Phys. Rev.* **52**, 269 (1937).

⁴ R. H. Fowler, *Statistical Mechanics*, 2d edition (Macmillan, 1936), pp. 36-46.

one can find the values of the quantities mentioned above using only the generating function, without finding the actual distribution.

DETERMINATION OF THE GENERATING FUNCTION

The method of determining the generating function, Q , will be given for a case somewhat more general than Eq. (4), in order that the results may be applied to the problems to be taken up later.

In general one has to solve a linear partial differential equation of the form

$$\partial Q/\partial t = X(\partial Q/\partial x) + Y(\partial Q/\partial y) + \dots, \quad (8)$$

where $X, Y, \dots$ are functions of $x, y, \dots$, subject to the boundary condition

$$Q(0, x, y, \dots) = F(x, y, \dots). \quad (9)$$

To such a partial differential equation is associated the set of ordinary differential equations⁵

$$\begin{aligned} d\xi/dt &= X(\xi, \eta, \dots), \\ d\eta/dt &= Y(\xi, \eta, \dots), \\ &\dots \end{aligned} \quad (10)$$

Their general solution is of the form

$$\begin{aligned} \xi &= f_1(t, x, y, \dots), \\ \eta &= f_2(t, x, y, \dots), \\ &\dots \end{aligned} \quad (11)$$

where $x, y, \dots$ appear as constants of integration, and the f 's are chosen so that

$$\begin{aligned} x &= f_1(0, x, y, \dots), \\ y &= f_2(0, x, y, \dots), \\ &\dots \end{aligned}$$

If the solution of Eq. (10) is known, that of Eqs. (8) and (9) is⁶

$$\begin{aligned} Q(t, x, y, \dots) \\ = F[f_1(t, x, y, \dots), f_2(t, x, y, \dots), \dots], \end{aligned} \quad (12)$$

which is the solution required.

APPLICATION TO THE PROBLEM OF PAIR PRODUCTION

One can now apply the results of the last section to the solution of Furry's Problem A.

⁵ Courant and Hilbert, *Methoden der Mathematischen Physik*, Vol. II (Julius Springer, 1927), p. 25.

⁶ Heinrich Weber, *Die Partiellen Differential Gleichungen der Mathematischen Physik* (Friedrich Vieweg und Sohn, 1900), p. 147.

From Eqs. (4), (8), and (10)

$$d\xi/dt = \xi^2 - \xi.$$

Integrating and setting $\xi = x$ at $t = 0$:

$$\xi = xe^{-t}/(xe^{-t} - x + 1) = e^{-t} \sum_{n=0}^{\infty} x^n (1 - e^{-t})^{n-1}. \quad (13)$$

The boundary condition $Q(x, 0) = x$ requires that $Q = \xi$, and comparing Eq. (13) with (3) it is evident that

$$P(n, t) = e^{-t}(1 - e^{-t})^{n-1}.$$

From Eqs. (6) and (13), the mean number of particles at depth t is

$$\langle n \rangle = e^t.$$

Therefore

$$P(n, t) = \langle n \rangle^{-1} [1 - \langle n \rangle^{-1}]^{n-1},$$

which is the solution obtained by Furry. From Eq. (7), the mean square is

$$\langle n^2 \rangle = 2e^{2t} - e^t.$$

The square of the standard deviation is then

$$\sigma_n^2 = \langle n^2 \rangle - \langle n \rangle^2 = e^t(e^t - 1).$$

In a Poisson distribution σ_n^2 is proportional to the mean number; in this problem it is roughly proportional to the square of the mean number, a substantial difference.

Another problem treated by Furry, Problem B, is stated as follows: The probability that in dt an electron produces a photon is dt , the electron itself continuing its course; the probability that in dt a photon produces a pair of electrons is $r dt$, the photon being absorbed. If one electron enters a sheet of thickness t , what is the probability, $P(n, t)$ that n electrons emerge?

Defining $p(n, m, t)$ as the probability that n electrons and m photons emerge, one has the differential equations

$$\begin{aligned} (d/dt)p(n, m, t) &= -(n + mr)p(n, m, t) \\ &+ np(n, m-1, t) + (m+1)rp(n-2, m+1, t) \end{aligned} \quad (14)$$

and the additional relation

$$P(n, t) = \sum_{m=0}^{\infty} p(n, m, t).$$

Defining the generating function

$$Q(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} x^n y^m p(n, m, t),$$

one gets the equation

$$\partial Q / \partial t = x(y-1)(\partial Q / \partial x) + r(x^2-y)(\partial Q / \partial y)$$

and from Eq. (10)

$$\begin{aligned} d\xi/dt &= \xi(1-\eta), \\ d\eta/dt &= r(\eta-\xi^2). \end{aligned}$$

These equations are not readily integrable so that in this case the method fails to give a useful result.

However, one can easily solve a problem that is essentially equivalent to the case $r = \infty$. It may be formulated as follows: The probability that an electron creates a pair in thickness dt is just dt , the electron continuing its course. If one electron enters a sheet of thickness t , what is the probability $P(n, t)$ that n particles emerge? Then the system of differential equations is

$$(d/dt)P(n, t) = -nP(n, t) + (n-2)P(n-2, t),$$

with the generating function

$$Q(x, t) = \sum_{n=0}^{\infty} x^n P(n, t).$$

One obtains, as before

$$\partial Q / \partial t = (x^3 - x)(\partial Q / \partial x)$$

and

$$d\xi/dt = \xi^3 - \xi.$$

Integrating by partial fractions

$$\begin{aligned} \xi &= xe^{-t} [1 - x^2(1 - e^{-2t})]^{-1/2}, \\ &= e^{-t} \sum_{n=0}^{\infty} (-1)^n \binom{-\frac{1}{2}}{n} x^{2n+1} (1 - e^{-2t})^n, \\ &= \sum_{n=0}^{\infty} x^{2n+1} P(2n+1, t), \end{aligned}$$

where $\binom{-\frac{1}{2}}{n}$ is the binomial coefficient. Therefore

$$P(2n+1, t) = (-1)^n \binom{-\frac{1}{2}}{n} e^{-t} (1 - e^{-2t})^n;$$

from the conditions of the problem, $P(2n, t) = 0$. The mean number

$$\langle n \rangle = e^{2t},$$

and therefore

$$P(2n+1, t) = (-1)^n \binom{-\frac{1}{2}}{n} (\langle n \rangle)^{-\frac{1}{2}} [1 - (\langle n \rangle)^{-1}]^n.$$

The mean square and square of the standard deviation are

$$\begin{aligned} \langle n^2 \rangle &= 3e^{4t} - 2e^{2t}, \\ \sigma_n^2 &= 2e^{2t}(e^{2t} - 1). \end{aligned}$$

APPLICATION TO SIMPLE RADIOACTIVE DECAY

As a first example of a somewhat different type of problem, consider the decay of a radioactive substance whose decay constant is λ . If $\langle n \rangle$ is the probable number of atoms present at time t , and ν the probable number at $t=0$,

$$(d/dt)(\langle n \rangle) = -\lambda \langle n \rangle, \quad \langle n \rangle = \nu e^{-\lambda t}.$$

To study the fluctuations of n about the probable value, we need to determine $P(n, t)$, the probability that there are just n atoms present at time t . These functions satisfy the differential equations,

$$(d/dt)P(n, t) = -\lambda n P(n, t) + \lambda(n+1)P(n+1, t). \quad (15)$$

To determine them completely their value at $t=0$, $P(n, 0)$, must be known. The generating function for Eq. (15) satisfies the equations

$$\partial Q / \partial t = \lambda(1-x)(\partial Q / \partial x),$$

$$Q(x, 0) = \sum_{n=0}^{\infty} x^n P(n, 0).$$

The appropriate solution of the associated ordinary differential equation is

$$\xi = (x-1)e^{-\lambda t} + 1,$$

and therefore

$$Q(x, t) = \sum_{n=0}^{\infty} \xi^n P(n, 0).$$

In general, the values of $P(n, 0)$ will not be

known; however, one may assume that $P(n, 0)$ has a maximum for some value of n approximately equal to ν . If this maximum is sharp and ν is large, the generating function may be approximated by

$$\begin{aligned} Q(x, t) &= \xi^\nu \\ &= [1 + e^{-\lambda t}(x-1)]^\nu \\ &= (a + bx)^\nu, \end{aligned}$$

where $b = 1 - a = e^{-\lambda t}$.

From this, it readily follows that

$$P(n, t) = \binom{\nu}{n} (1-b)^{\nu-n} b^n. \quad (16)$$

The value of $\langle n \rangle$, obtained by differentiating Q , is found to be νb . Differentiating again,

$$\langle n^2 \rangle - \langle n \rangle = \nu(\nu-1)b^2,$$

from which $\sigma_n^2 = \langle n \rangle a$.

For large values of t , a approaches 1 and the Poisson formula, $\sigma_n^2 = \langle n \rangle$, is valid.

However, one is usually not interested in $\langle n \rangle$ or σ_n^2 , but in the number of atoms that decay in the time interval from t to $t + \Delta$. To discuss the fluctuations in the number decaying in this interval, one requires the probability that there are n atoms at time t , and m at time $t + \Delta$. Let $P(\nu, n, t)$ be the function previously called $P(n, t)$ and defined by Eq. (16); then the required probability is easily seen to be

$$p(n, m) = P(\nu, n, t)P(n, m, \Delta),$$

which has the generating function

$$Q(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p(n, m) x^n y^m.$$

From Eq. (16)

$$Q(x, y, t) = \sum_{n=0}^{\infty} P(\nu, n, t) x^n (\alpha + \beta y)^\nu$$

$$= [a + bx(\alpha + \beta y)]^\nu,$$

where $\beta = 1 - \alpha = e^{-\lambda \Delta}$.

The probability that there are just s disintegra-

tions in the interval is

$$p(s) = \sum_{m=0}^{\infty} p(m+s, m),$$

with the generating function

$$\begin{aligned} Q(x, 1/x, t) &= [a + b(\alpha x + \beta)]^\nu \\ &= [A + Bx]^\nu. \end{aligned}$$

where

$$A = a + b\beta, \quad B = b\alpha = 1 - A.$$

Consequently,

$$p(s) = \binom{\nu}{s} A^{\nu-s} B^s,$$

$$\langle s \rangle = \nu B = \langle n \rangle_t - \langle n \rangle_{t+\Delta},$$

$$\langle s^2 \rangle = (\langle s \rangle)^2 + \langle s \rangle A,$$

$$\sigma_s^2 = \langle s \rangle A.$$

In case $B \ll 1$ and $\nu \gg 1$ the generating function approximates

$$Q(x, 1/x, t) = \exp [\langle s \rangle (x-1)]$$

and $p(s) = (1/s!) \exp [-\langle s \rangle] (\langle s \rangle)^s$,

which is the Poisson formula. These results are not new, but by applying the same methods to chain disintegrations, new results are obtainable.

CHAIN DISINTEGRATIONS

Suppose one has three substances A , B , and C , such that A disintegrates with decay constant λ_1 to form B , and B disintegrates with decay constant λ_2 to form C , a stable element. Let $\langle k \rangle$, $\langle m \rangle$, $\langle n \rangle$, be the probable number of each present at time t . Then

$$(d/dt) \langle k \rangle = -\lambda_1 \langle k \rangle,$$

$$(d/dt) \langle m \rangle = \lambda_1 \langle k \rangle - \lambda_2 \langle m \rangle, \quad (17)$$

$$(d/dt) \langle n \rangle = \lambda_2 \langle m \rangle.$$

The solution of these equations is well known and is

$$\langle k \rangle = \kappa a(t),$$

$$\langle m \rangle = \kappa c(t) + \mu b(t),$$

$$\langle n \rangle = \kappa e(t) + \mu d(t) + \nu,$$

where

$$a = e^{-\lambda(1)t}, \quad b = e^{-\lambda(2)t}, \quad c = [\lambda_1/(\lambda_2 - \lambda_1)](a - b), \\ d = 1 - b, \quad e = 1 - a - c,$$

and κ, μ, ν are the probable numbers of each present at $t=0$.

To study the fluctuations about the average, one must find the probability that at time t there are k, m, n of the three kinds present. The probabilities satisfy the system of differential equations⁷

$$(d/dt)P(k, m, n) = -k\lambda_1 P(k, m, n) \\ + (k+1)\lambda_1 P(k+1, m-1, n) - m\lambda_2 P(k, m, n) \\ + (m+1)\lambda_2 P(k, m+1, n-1). \quad (18)$$

To solve these equations completely one has to have the values of $P(k, m, n)$ at $t=0$, which are usually not given. However, we shall assume that, at $t=0$, $P(k, m, n)$ has a sharp maximum at $k=\kappa, m=\mu, n=\nu$. Then the generating function for Eq. (18) is approximately given by the boundary condition

$$Q(x, y, z, 0) = x^\kappa y^\mu z^\nu.$$

From Eq. (18) one obtains

$$\partial Q/\partial t = \lambda_1(y-x)(\partial Q/\partial x) + \lambda_2(z-y)(\partial Q/\partial y), \quad (19)$$

with the associated ordinary differential equations

$$d\xi/dt = \lambda_1(\eta - \xi), \\ d\eta/dt = \lambda_2(\zeta - \eta), \\ d\zeta/dt = 0.$$

Integrating directly, one gets (using the same definitions of $a, \dots, e$ as above)

$$\xi = ax + cy + ez, \\ \eta = by + dz, \\ \zeta = z,$$

from which

$$Q(x, y, z, t) = (ax + cy + ez)^\kappa (by + dz)^\mu z^\nu. \quad (20)$$

From the definition of the generating function, one can get $P(k, m, n)$ by picking out the coefficients of $x^k y^m z^n$ in Eq. (20).

The probability, $P(K, t)$ that there are K atoms of kind A regardless of the number of

kind B and C is given by the coefficient of x^K in

$$Q(x, 1, 1, t) = (ax + c + e)^\kappa,$$

from which

$$P(K, t) = \binom{\kappa}{K} (1-a)^{\kappa-K} a^K. \quad (21)$$

Eq. (21) is in the same form as Eq. (16) and therefore the fluctuations in the number of disintegrations $A \rightarrow B$ will be the same as for the simple radioactive case. This is not true for disintegrations $B \rightarrow C$.

The probability, $P(M, t)$ that there are M atoms of kind B at time t is the coefficient of y^M in

$$Q(1, y, 1, t) = (a + cy + e)^\kappa (by + d)^\mu. \quad (22)$$

The $P(M, t)$ cannot be calculated without difficulty except for the case $\mu=0$. However $\langle M \rangle$, $\langle M^2 \rangle$, and σ_M are easy to determine by differentiation of Eq. (22). The probability $P(N, t)$ for atoms of kind C can be treated in a similar manner.

To study the fluctuations in the number of disintegrations, let $P(\kappa, \mu, \nu; k, m, n; t)$ be the probability of k, m, n atoms of each kind at time t , given κ, μ, ν at $t=0$. Then one can define the function

$$p(k, m, n; k', m', n') \\ = P(\kappa, \mu, \nu; k, m, n; t) P(k, m, n; k', m', n'; \Delta)$$

as the probability that in an interval $t, t+\Delta$, the number of atoms of each kind has changed from k, m, n to k', m', n' . The $p(k, m, n; k', m', n')$ are generated by

$$Q(x, y, z; u, v, w)$$

$$= \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} P(\kappa, \mu, \nu; k, m, n) x^k y^m z^n$$

$$\times [\alpha u + \gamma v + \epsilon w]^k [\beta v + \delta w]^\mu z^\nu$$

$$= [ax(\alpha u + \gamma v + \epsilon w) + cy(\beta v + \delta w) + ez w]^\kappa$$

$$\times [by(\beta v + \delta w) + dz w]^\mu (zw)^\nu,$$

where $a, \dots, e$ are as before, and

$$\alpha = e^{-\lambda(1)\Delta}, \quad \beta = e^{-\lambda(2)\Delta}, \quad \gamma = [\lambda_1/(\lambda_2 - \lambda_1)](\alpha - \beta),$$

$$\delta = 1 - \beta, \quad \epsilon = 1 - \alpha - \gamma.$$

⁷ The validity of Eq. (18) may be checked by deriving Eq. (17) from them, using the usual definition of k, m, n in terms of $P(k, m, n)$.

The probability that $k = k' + r$, $n = n' - s$, with k, m, n, m' anything whatever is generated by

$$Q(x, 1, 1/w; 1/x, 1, w) = [ax(\alpha u + \gamma v + \epsilon w) + c(\beta + \delta w) + e]^\kappa \times [b(\beta + \delta w) + d]^\mu \\ = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} p(r, s) x^r w^s \quad (23)$$

if $p(r, s)$ is the probability of r disintegrations of kind $A \rightarrow B$, and s of kind $B \rightarrow C$.

The probability of r disintegrations of kind $A \rightarrow B$, is

$$p'(r) = \sum_{s=0}^{\infty} p(r, s).$$

From Eq. (23), with $w = 1$,

$$\sum_{r=0}^{\infty} x^r p'(r) = [A' + B'x]^\kappa, \quad (24)$$

where

$$A' = a\alpha + c + e, \quad B' = a(1 - \alpha), \quad A' + B' = 1.$$

This $p'(r)$ is then the coefficient of x^r in the expansion of Eq. (24), in agreement with the remark made above. The mean number and standard deviation, from Eq. (24), turns out to be

$$\langle r \rangle = \kappa a(1 - \alpha) = \kappa e^{-\lambda(1)} t(1 - e^{-\lambda(1)\Delta}), \\ \sigma_r^2 = (\langle r \rangle)[1 - (\langle r \rangle)/\kappa].$$

The probability of s disintegrations of kind $B \rightarrow C$ is

$$p''(s) = \sum_{r=0}^{\infty} p(r, s).$$

From Eq. (23), with $x = 1$,

$$\sum_{s=0}^{\infty} w^s p''(s) = [A'' + B''w]^\kappa [C'' + D''w]^\mu,$$

where

$$A'' = a(1 - \epsilon) + c\beta + e, \quad B'' = a\epsilon + \delta c, \quad C'' = b\beta + d, \\ D'' = \delta b, \quad A'' + B'' = C'' + D'' = 1.$$

One obtains easily

$$\langle s \rangle = \kappa(a\epsilon + c\delta) + \mu b\delta, \\ \sigma_s^2 = \langle s \rangle - \kappa(a\epsilon + c\delta)^2 - \mu(b\delta)^2.$$

If the disintegration products are indistinguishable it is of interest to find the fluctuations in the total number of disintegrations. Setting $x = w$ in Eq. (23), one obtains for the probability of s disintegrations of either kind

$$p'''(s) = \sum_{r=0}^{\infty} p(s - r, r),$$

which is the coefficient of x^s in the expansion of

$$\sum_{s=0}^{\infty} x^s p'''(s) = [A''' + C'''x + E'''x^2]^\kappa [B''' + D'''x]^\mu,$$

where

$$A''' = a\alpha + c\beta + e, \quad B''' = b\beta + d, \quad C''' = a\gamma + c\delta, \\ D''' = b\delta, \quad E''' = a\epsilon, \\ A''' + B''' + C''' = B''' + D''' = 1.$$

By differentiation, one obtains

$$\langle s \rangle = \kappa(C''' + 2E''') + \mu D''', \\ \sigma_s^2 = \langle s \rangle + \kappa E'''(C''' + 2A''') - \mu D'''^2.$$

These formulae are quite elaborate, analytically, but are not difficult for numerical evaluation.

BRANCH DISINTEGRATIONS

Suppose an atom A can disintegrate either with decay constant λ_1 to form B , or with decay constant λ_1' to form C , and that B and C disintegrate with decay constants λ_2 and λ_3 to form D , a stable element. The probable number of atoms of each satisfy the equations

$$(d/dt)(\langle h \rangle) = -(\lambda_1 + \lambda_1')\langle h \rangle, \\ (d/dt)(\langle k \rangle) = \lambda_1\langle h \rangle - \lambda_2\langle k \rangle, \\ (d/dt)(\langle m \rangle) = \lambda_1'\langle h \rangle - \lambda_3\langle m \rangle, \\ (d/dt)(\langle n \rangle) = \lambda_2\langle k \rangle + \lambda_3\langle m \rangle,$$

with the solutions

$$\langle h \rangle = a\theta, \\ \langle k \rangle = b\theta + e\kappa, \\ \langle m \rangle = c\theta + g\mu, \\ \langle n \rangle = d\theta + f\kappa + j\mu + \nu,$$

where

$$a = e^{-(\lambda_1 + \lambda_1')t}$$

$$b = [\lambda_1 / (\lambda_2 - \lambda_1 - \lambda_1')](a - e),$$

$$c = [\lambda_1' / (\lambda_3 - \lambda_1 - \lambda_1')](a - g),$$

$$a + b + c + d = e + f = g + j = 1,$$

and θ, κ, μ, ν are the probable numbers of each present at $t=0$. Just as in the previous problem, one can define

$$Q(w, x, y, z, t) = \sum_{h=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} w^h x^k y^m z^n P(h, k, m, n)$$

and obtain the equation

$$\begin{aligned} \partial Q / \partial t = & (\partial Q / \partial w)(\lambda_1 x + \lambda_1' y - \lambda_1 w - \lambda_1' w) \\ & + (\partial Q / \partial x)\lambda_2(z - x) + (\partial Q / \partial y)\lambda_3(z - y), \end{aligned}$$

to be solved subject to

$$Q(w, x, y, z, 0) = w^\theta x^\kappa y^\mu z^\nu.$$

The associated system of ordinary differential equations is

$$d\psi/dt = \lambda_1 \xi + \lambda_1' \eta - (\lambda_1 + \lambda_1')\psi,$$

$$d\xi/dt = \lambda_2(\zeta - \xi),$$

$$d\eta/dt = \lambda_3(\zeta - \eta),$$

$$d\zeta/dt = 0.$$

The value of Q turns out to be

$$\begin{aligned} Q(w, x, y, z, t) = & (aw + bx + cy + dz)^\theta \\ & \times (ex + fz)^\kappa (gy + iz)^\mu z^\nu, \end{aligned}$$

from which one can calculate the fluctuations in the number of atoms present and in the number of disintegrations as before. There is such a variety of possible calculations that it scarcely seems worth while to make any without having some specific experiment in mind.

CONCLUDING REMARKS

It has been the aim of the writer to develop a method for the solution of fluctuation problems that cannot be handled by ordinary methods such as the assumption of a Poisson distribution. The method consists of finding the system of differential equations that are satisfied by the probabilities under consideration, the definition of a generating function for these probabilities and the determination of the generating function from the differential equations. The method has been shown to be effective in the solution of a variety of problems and is so general that there are undoubtedly many others that can be solved by its use.

An important feature of the method is that many useful quantities such as the standard deviation and the mean of the distribution can be found from the generating function in those cases where the distribution itself is difficult to work out.

The writer wishes to express his sincere appreciation to Dr. Carl Eckart for suggesting the problem and for continued interest and advice in the progress of its solution.

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